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# On the number of k-matchings of graphs

**F. Asgari[1], R. Namazi[1], R. Vesalian[1*] and M. Zallaghi[2]**
[1]Department of Mathematics, Mahshahr Branch, Islamic Azad University, Mahshahr, Iran
[2]Department of Mathematics, Isfahan University, Isfahan, Iran
r.vesalian@yahoo.com

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## Abstract

*In this paper an inductive formula for the number of k-matchings in graphs is derived using this formula. We concluded the number of k-matchings in special regular graphs and complete graphs.*

**Keywords:** k-matching, matching polynomial, regular graphs.

## Introduction

Let $G = (V, E)$ be graph in which $V(G)$ and $E(G)$ are the numbers of vertices and edges respectively. A matching in graph $G$ is by definition a spanning sub graph of $G$ whose components are vertices and edges. A k-matching is a matching with edges only. We show the number of k-matchings in a graph $G$ by $P(G, K)$ and assume $P(G, 0) = 1$.

Based on matching in a graph $G$ we define the matching polynomial $\mu(G, x)$ as follow

$$\mu(G,x) = \sum_{k=0}^{[n/2]} (-1)^k P(G,K) x^{n-2k}$$

In which$n$ is the number of vertices of graph $G$.

We note that the graphs here are finite, loop less and contain no multiple edges.

The matching polynomial can be a tool for characterization of graphs. Two isomorphic graphs have the same matching polynomials that are called co-matching graphs.

However two co-matching graphs are not necessarily isomorphic[1].

## Preliminaries

Finding the number of k-matching for $k = 0,1, \dots ,6$ have been done so far. For example it is easy to see $P(G, 1) = m$ in which $m$ is the number of edges.

For the number of two and three matching we have [2],

$$P(G,2) = \binom{m}{2} - \sum_{i=1}^{n} \binom{d_i}{2}$$

$$P(G,3) = \binom{m}{3} - (m-2)\sum_{i}\binom{d_i}{2} + 2\sum_{i}\binom{d_i}{3} + \sum_{ij}(d_i - 1)(d_j - 1) - N_T$$

In which $N_T$ is the number of triangles in $G$.

The number of k-matchnigs for $k = 4,5,6$ can be found in literatures[2-10].

The number of k-matchings calculated in the mentioned works shows when $k$ grow up the formula for the number of k-matching gets very long and complicated. So calculating this number for $k \geq 7$ directly is not so logical and practical. Therefore in this work we derive an inductive formula for the number of k-matchings that makes it much easier to find it.

## Number of k-matchings

Theorem 3.1: let $G$ be a simple graph of order $n$ and $E(G)$ be the set of it's edges. Then the number of k-matchings in graph $G$ is:

$$P(G,k) = \frac{1}{k} \sum_{ij \epsilon E(G)} P(G - i - j, k - 1)$$

Proof: let $S(G, k)$ be the set of all k-matchings in $G$. We consider an orbitary edge $ij$ from $E(G)$ then we have two cases:

Case I: $ij$ is not the component of any k-matchings in $S(G, k)$ therefore $P(G - i - j, k - 1) = 0$.

Case II: $ij$ is not the component of at least one of the k-matchings in $S(G, k)$ so the number of matchings in $S(G, k)$ such that $ij$ is one of their components is $P(G - i - j, k - 1)$

Now according to above cases by choosing any of k-matching in $S(G, k)$, this k-matching is counted $k$ times so:
$P(G,k) = \frac{1}{k} \sum_{ij \epsilon E(G)} P(G - i - j, k - 1)$

Corollary 3.2: if $G$ is a simple graph then:

$$P(G,k) = \frac{1}{k!}\sum_{i_1j_1}\sum_{i_2j_2}\dots\sum_{i_kj_k}1$$

In which the edges $i_1j_1, i_2j_2, \dots, i_kj_k$ changes in the sets of edges of $E(G), E(G-i_1-j_1), \dots, E(G-i_1-j_1-\dots-i_{k-1}-j_{k-1})$ respectively.

Proof: according to theorem 3.1:

$$P(G,k) = \frac{1}{k}\sum_{i_1j_1\epsilon E(G)} P(G-i_1-j_1, k-1)$$

And again using the above formula for graph $G-i_1-j_1$ we have:

$$P(G-i_1-j_1,k-1) = \frac{1}{k-1}\sum_{i_2j_2\epsilon E(G-i_1-j_1)} P(G-i_1-j_1-i_2-j_2, k-2)$$

So

$$P(G,k) = \frac{1}{k(k-1)}\sum_{i_1j_1}\sum_{i_2j_2} P(G-i_1-j_1-i_2-j_2,k-2)$$

So after k times:

$$P(G,k) = \frac{1}{k(k-1)\dots(1)}\sum_{i_1j_1}\sum_{i_2j_2}\dots\sum_{i_kj_k} P(G-i_1-j_1-\dots-i_k-j_k,0)$$

But

$$P(G-i_1-j_1-\dots-i_k-j_k,0) = 1$$

And the theorem is proved.

Example: let $G$ be a connected, 3-regular graph of order 8 (Figure-1), we calculate $P(G,4)$

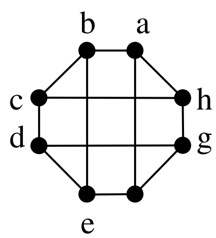



**Figure-1**

According to result 3.2:

$$P(G,4) = \frac{1}{4!}\sum_{i_1j_1}\sum_{i_2j_2}\sum_{i_3j_3}\sum_{i_4j_4}1$$

In which $i_1j_1\epsilon E(G), i_2j_2\epsilon E(G-i_1-j_1), i_3j_3\epsilon E(G-i_1-j_1-i_2-j_2)$ and $i_4j_4\epsilon E(G-i_1-j_1-i_2-j_2-i_3-j_3)$.

Now if $i_1j_1\epsilon E(G)$ be any of edges, $ab, bc, cd, de, ef, fg, gh, ha, af, be, ch, dg$ then the graph $G-i_1-j_1$ will be isomorphic with graph $H$ (Figure-2):

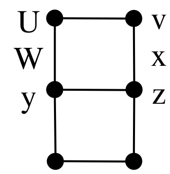



**Figure-2**

So

$$P(G,4) = \frac{12}{4!}\sum_{i_2j_2}\sum_{i_3j_3}\sum_{i_4j_4}1$$

In which $i_2j_2\epsilon E(H), i_3j_3\epsilon E(H-i_2-j_2), i_4j_4\epsilon E(H-i_2-j_2-i_3-j_3)$ for $i_2j_2\epsilon E(H)$ we consider three following cases:

CaseI: If $i_2j_2$ belonges to the set of edges $E_1 = \{uv, ef\}$ then the graph $H-i_2-j_2$ is isomorphic with graph $M$ (Figure-3):

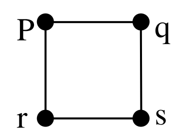



**Figure-3**

So

$$\sum_{i_2j_2\epsilon E_1}\sum_{i_3j_3}\sum_{i_4j_4}1 = 2\sum_{i_3j_3}\sum_{i_4j_4}1$$

In which $i_3j_3\epsilon E(M)$ and $i_4j_4\epsilon E(M-i_3-j_3)$.

Now because $i_3j_3\epsilon E(M)$ therefor $M-i_3-j_3$ will be isomorphic with single edged graph (Figure-4)

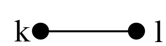



**Figure-4**

So

$$\sum_{i_3j_3\epsilon E(M)}\sum_{i_4j_4}1 = 4\sum_{i_4j_4}1\ (i_4j_4 = kl) = 4$$

Therefore

$$\sum_{i_2j_2\epsilon E_1}\sum_{i_3j_3}\sum_{i_4j_4}1 = 2\sum_{i_3j_3}\sum_{i_4j_4}1 = 2\times 4 = 8$$

CaseII: If $i_2j_2$ belongs to the set of edges $E_2 = \{uw, vx, wy, xz\}$ then graph, $H-i_2-j_2$isomorphic with $N$ (Figure-5)

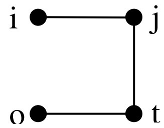


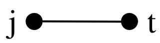

**Figure-5**

so

$$\sum_{i_2j_2\epsilon E_3}\sum_{i_3j_3}\sum_{i_4j_4}1 = 4\sum_{i_3j_3}\sum_{i_4j_4}1$$

in which $i_3j_3\epsilon E(N)$ and $i_4j_4\epsilon E(N - i_3 - j_3)$.

If $i_3j_3$ belongs to set of edges $E_2^{'} = \{ij, ot\}$ then $N - i_3 - j_3$ is isomorphic with following single edged graph:




So

$$\sum_{i_3j_3\epsilon E_2^{'}}\sum_{i_4j_4}1 = 2\sum_{i_4j_4}1\ (i_4j_4 = jt) = 2$$

But if $i_3j_3 = jt$ then $N - i_3 - j_3$ will be isomorphic with the following null graph:


And so there is no choice for $i_4j_4$ . Therefore

$$\sum_{i_3j_3=jt}\sum_{i_4j_4}1 = 0$$

Consequently in this case we have:

$$\sum_{i_2j_2\epsilon E_3}\sum_{i_3j_3}\sum_{i_4j_4}1 = 4\sum_{i_3j_3}\sum_{i_4j_4}1$$
$$= 4\left(\sum_{i_3j_3\epsilon E_2^{'}}\sum_{i_4j_4}1 + \sum_{i_3j_3=jt}\sum_{i_4j_4}1\right)$$
$$= 4(2 + 0) = 8$$

Case III: If $i_2j_2 = wx$ then the graph $H - i_2 - j_2$ is isomorphic with graph $R$ (Figure-6)

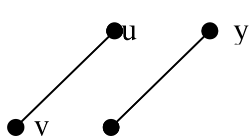



**Figure-6:**

So

$$\sum_{i_2j_2=wx}\sum_{i_3j_3}\sum_{i_4j_4}1 = \sum_{i_3j_3}\sum_{i_4j_4}1$$

In which $i_3j_3\epsilon E(R), i_4j_4\epsilon E(R - i_3 - j_3)$

Now since $i_3j_3\epsilon E(R)$ therefor graph $R - i_3 - j_3$ is isomorphic with following single edged graph (Figure-7)

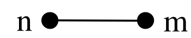



**Figure-7**

So

$$\sum_{i_3j_3\epsilon E(R)}\sum_{i_4j_4}1 = 2\sum_{i_4j_4}1\ (i_4j_4 = mn) = 2$$

Therefore

$$\sum_{i_2j_2=wx}\sum_{i_3j_3}\sum_{i_4j_4}1 = 2\sum_{i_3j_3\epsilon E(R)}\sum_{i_4j_4}1 = 2$$

Finally:

$$P(G,4) = \frac{12}{4!}\sum_{i_2j_2}\sum_{i_3j_3}\sum_{i_4j_4}1$$
$$= \frac{12}{4!}\left(\sum_{i_2j_2\epsilon E_1}\sum_{i_3j_3}\sum_{i_4j_4}1 + \sum_{i_2j_2\epsilon E_3}\sum_{i_3j_3}\sum_{i_4j_4}1 + \sum_{i_2j_2=wx}\sum_{i_3j_3}\sum_{i_4j_4}1\right)$$
$$= \frac{12}{4!}(8 + 8 + 2) = 9$$

Corollary 3.3: if $G$ be the $2^P$ regular graph of order $2^{P+1}$ then if$k \leq 2^P + 1$ :

$$P(G,K) = \frac{1}{k!}\prod_{S=1}^{k}(2^P - S + 1)^2$$

Proof: let $m(G)$ be the number of edges. Because $G$ is a $2^P$ regular graph of order $2^{P+1}$ so $m(G) = 2^{2P}$

We assume $G_1 = G$ and choose the edge $i_1j_1$ from $G_1$ the graph $G_2 = G_1 - i_1 - j_1$ will be of order $2^{P+1} - 2$. Since the vertices $i_1$ and $j_1$ except each other are connected to $2^P - 1$ other vertices so if we omit the the vertices $i_1, j_1$ from graph $G$, then the $(2^P - 1) + (2^P - 1) = (2^{P+1} - 2)$ vertices of graph $G_2$ are all of degree $2^P - 1$. This means the graph $G_2$ is a $2^P - 1$ regular graph of order $2^{P+1} - 2$ .threrfore

$$m(G_2) = \frac{1}{2}(2^{P+1} - 2)(2^P - 1) = (2^P - 1)^2$$

Preceding this approach and using the same method. If we consider the edge $i_2j_2$ from $(2^P - 1)^2$ edges of graph $G_2$, the graph $G_3 = G_2 - i_2 - j_2$ is $(2^P - 1)^2$ regular and of order $2^{P+1} - 4$ and therefor:

$$m(G_3) = (2^P - 2)^2$$

After $k$ steps, with induction we deduce that the graph $G_k = G_{k-1} - i_{k-1} - j_{k-1}$ is $2^P - k + 1$ regular of order $2^{P+1} - 2k +$

2 and so $m(G_k) = (2^P - k + 1)^2$ but $2^P - k + 1 \geq 0$ that means $k \leq 2^P + 1$.

Now using the corollary 2.3 we have:

$$P(G,k) = \frac{1}{k!} \sum_{i_1 j_1 \epsilon E(G_1)} \sum_{i_2 j_2 \epsilon E(G_2)} \dots \sum_{i_k j_k \epsilon E(G_k)} 1$$

$$= \frac{1}{k!} m(G_1) m(G_2) \dots m(G_k)$$

$$= \frac{1}{k!} \prod_{S=1}^{k} m(G_S)$$

$$= \frac{1}{k!} \prod_{S=1}^{k} (2^P - S + 1)^2$$

Corollary 3.4: if $G$ is a complete graph of order $n$ then with assumption $k \leq \frac{n+1}{2}$:

$$P(G,k) = \frac{n!}{2^k . k! (n-2k)!}$$

Proof: if $G$ is a complete graph of order $n$ then the degree of any vertex of $G$ is $n-1$ and it's size is $\binom{n}{2}$. Assuming $G_1 = G$ and choosing the edge $i_1 j_1$ from $G_1$ the graph $G_2 = G_1 - i_1 - j_1$ is a complete graph of order $n-2$ and so it's size is $\binom{n-2}{2}$. Therefore by induction we conclude that the graph $G_k = G_{k-1} - i_{k-1} - j_{k-1}$ is a graph of order $n - 2k + 2$ and size $\binom{n-2k+2}{2}$.

But because have the degree of the vertices of $G_k$ is $n - 2k + 2$ so $n - 2k + 2 \geq 0$ or equivalently $\leq \frac{n+1}{2}$.

Now according to corollary 3.2

$$P(G,k) = \frac{1}{k!} \sum_{i_1 j_1 \epsilon E(G_1)} \sum_{i_2 j_2 \epsilon E(G_2)} \dots \sum_{i_k j_k \epsilon E(G_k)} 1$$

$$= \frac{1}{k!} m(G_1) m(G_2) \dots m(G_k)$$

$$= \frac{1}{k!} \binom{n}{2} \binom{n-2}{2} \dots \binom{n-2k+2}{2}$$

$$= \frac{n!}{2^k . k! (n-2k)!}$$

## Conclusion

The result of this paper shows that a recursive formula for finding the number of matching in a graph is more applicable than a direct computation as we see in our previous work the formulas for the number of six and seven matchings are really long and complicated.

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