The study of flow rate, resistive impedance of blood flowing through stenosed artery

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Abstract

The present paper aims to compute flow rate, resistive impedance of blood flowing through stenosed artery. The flow of blood in constricted artery is studied. The blood is treated as Newtonian fluid. The equations involved in the mathematical model are solved using finite difference approximations. The flow rate is calculated at the beginning and end of arterial segment. It is also calculated in the region of stenosis. Flow rate and resistive impedance are plotted axially for different values of time.

Keywords: Resistive impedance, Flow rate, Finite difference scheme.

Introduction

The basic functions of body parts of living organism are due to the flow of fluids. The proper functioning of many important organs of human body depends on blood flow. The shortage in bio-fluid transport at any scale can result in major cardiovascular diseases; these diseases have adverse effects on health. It has been main reason for deaths of many people worldwide. Therefore it is important to study the flow of blood in stenosed arteries of human body for better diagnostics.

Young D.F.[1] studied the impact of time dependent stenosis on flow through a tube. The paper discussed the effect of growth of stenosis on pressure distribution and wall shear stress. Stenosis is the deposition of fatty material inside arterial lumen. A sticky deposit is formed due to the interaction of blood cells with cells of the wall. These deposits normally occur where arteries have bends and at the branches where blood flow is distributed and a secondary flow is created[2]. Haldar K.[3] has studied resistance to arterial blood flow due to shape function. The paper investigated that the opposition to blood flow reduces with change in the shape of stenosis. They have considered the flow to be steady state one dimensional. The blood flow through an artery having overlapping stenosis was investigated by Shrivastava V.P. et. al[4]. They have derived expression for flow characteristics like resistance to flow, wall shear stress, the shear stress in the region of stenosis and critical height of the stenosis. Yakhot A. et. al[5] have studied a pulsatile flow of a viscous, incompressible fluid through a stenosed artery and discussed the influence of the shape and roughness of uppermost layer of arterial wall on the blood flow resistance. Flow rate, the rate of change of pressure at various locations of stenosed region and resistance to blood flow due to pulsatile flow in elastic carotid artery were calculated by Agarwal Ruchi et. al[6]. They examined in detail the

blood flow at different places in common carotid artery and internal carotid artery. The blood flow is calculated for different frequencies. They have treated carotid artery to be elastic.

In present study blood is treated as Newtonian fluid, the equations involved in the mathematical model are discretized using finite difference approximations. The MATLB simulation is used for numerical calculations. The flow rate distribution along the length of artery is discussed for different time. Flow rate and resistive impedance are plotted axially for different values of time.

Governing equations: The flow of blood is taken as unsteady, laminar and axially symmetric. The basic equations of motion governing such flow can be written from Mazumdar J.N.[7].

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \tag{1}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial r} + w\,\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial r} + \mu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{2}$$

$$\rho\left(\frac{\partial w}{\partial t} + u\,\frac{\partial w}{\partial r} + w\,\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right) \tag{3}$$

Here r and z are radial and axial directions. u and w are velocities in radial and axial directions. ρ and μ are density and viscosity of blood, p is the pressure. The pressure gradient $-\frac{\partial p}{\partial z}$ is given by[8] $-\frac{\partial p}{\partial z} = A_0 + A_1 \cos \omega t$

Where: A_0 is the constant amplitude and A_1 is the amplitude of the vibrating component which results into systolic and diastolic pressure. Here w = $2\pi f_p$, where f_p is pulse frequency.

The geometry of stenosis: The stenosis geometry is time dependent. Multiple stenosis regions are overlapped. It is described from Chakravarty S.[9] by

$$R(z,t) = a\left\{1 - \frac{\tau_m}{5005\,a\,l_0{}^6}\left[\frac{668662}{9}(z-d)\,l_0{}^5 - 370281(z-d)^2\,l_0{}^4 + 743344(z-d)^3\,l_0{}^3 - 698476(z-d)^4\,l_0{}^2 + 307584(z-d)^5 l_0 - 51264(z-d)^6\right]\right\}$$
$$\dots\dots d < z < d + 2l_0$$
$$= a a_1(t) \qquad \dots\dots\text{Otherwise}$$

Where: $a_1(t) = 1 - \cos(\omega t - 1)\,\beta e^{\beta \omega t}$ in which ω is angular frequency and β is the constant

Boundary conditions: The velocities at the inlet and outlet of an arterial segment of finite length are taken as Chakravarty S. et al[9].

It is assumed that initially radial and axial velocity both are zero. That is when system is at rest there is no flow through artery

i.e. $u(r,z,0) = 0, w(r,z,0) = 0$ (4)

Axially, there is no radial flow, therefore the radial velocity is zero and the axial velocity gradient of the blood can be taken as zero. This can be written as

$$\frac{\partial w}{\partial r} = 0, u(r,z,t) = 0 \quad \text{on } r = 0 \tag{5}$$

On the artery wall the axial velocity is zero due to no slip condition and radial velocity is rate of change in shape of the stenosis which is written as

$$w(r,z,t) = 0, u(r,z,t) = \frac{\partial R}{\partial t} \quad \text{on } r = R(z,t) \tag{6}$$

Methodology

Introducing radial co-ordinate transformation, $x = \frac{r}{R(z,t)}$, Equations (1), (3) and boundary conditions (4) - (6) take the following form

$$\frac{1}{R}\frac{\partial u}{\partial x} + \frac{u}{xR} + \frac{\partial w}{\partial z} - \frac{x}{R}\frac{\partial R}{\partial z}\frac{\partial w}{\partial x} = 0 \tag{7}$$

$$\frac{\partial w}{\partial t} = \frac{1}{R}\left[x\left(w\frac{\partial R}{\partial z} + \frac{\partial R}{\partial t}\right) - u\right]\frac{\partial w}{\partial x} - w\frac{\partial w}{\partial z} + \frac{\mu}{\rho R^2}\left(\frac{\partial^2 w}{\partial x^2} + \frac{1}{x}\frac{\partial w}{\partial x}\right) - \frac{1}{\rho}\frac{\partial p}{\partial z} \tag{8}$$

$$u(x,z,0) = 0, w(x,z,0) = 0 \tag{9}$$

$$\frac{\partial w}{\partial x} = 0, u(x,z,t) = 0 \quad \text{on } x = 0 \tag{10}$$

$$w(x,z,t) = 0, u(x,z,t) = \frac{\partial R}{\partial t} \quad \text{on } x = 1 \tag{11}$$

Solving equation (8) using finite difference approximations in which central differences have been used.

$$\frac{\partial w}{\partial x} = \frac{w_{i,j+1}^k - w_{i,j-1}^k}{2\Delta x}$$

$$\frac{\partial w}{\partial z} = \frac{w_{i+1,j}^k - w_{i-1,j}^k}{2\Delta z}$$

$$\frac{\partial w}{\partial t} = \frac{w_{i,j}^{k+1} - w_{i,j}^k}{\Delta t}$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{w_{i,j+1}^k - 2w_{i,j}^k + w_{i,j-1}^k}{(\Delta x)^2}$$

Where: $x_j = (j-1)\Delta x$, $z_i = (i-1)\Delta z$ and $t_k = (k-1)\Delta t$.

Here $\Delta x, \Delta z$ are increments in vertical and horizontal directions.

$$w_{i,j}^{k+1} = w_{i,j}^k + \Delta t\left\{\frac{-1}{\rho}\left(\frac{\partial p}{\partial z}\right)^{k+1} - w_{i,j}^k\left(\frac{w_{i+1,j}^k - w_{i-1,j}^k}{2\Delta z}\right) + \left(\frac{x_j}{R_i^k}w_{i,j}^k\left(\frac{\partial R}{\partial z}\right)_i^k + \frac{x_j}{R_i^k}\left(\frac{\partial R}{\partial t}\right)_i^k - \frac{u_{i,j}^k}{R_i^k}\right)\left(\frac{w_{i,j+1}^k - w_{i,j-1}^k}{2\Delta x}\right) + \frac{\mu}{\rho(R_i^k)^2}\left[\frac{w_{i,j+1}^k - 2w_{i,j}^k + w_{i,j-1}^k}{(\Delta x)^2} + \frac{1}{x_j}\left(\frac{w_{i,j+1}^k - w_{i,j-1}^k}{2\Delta x}\right)\right]\right\} \tag{12}$$

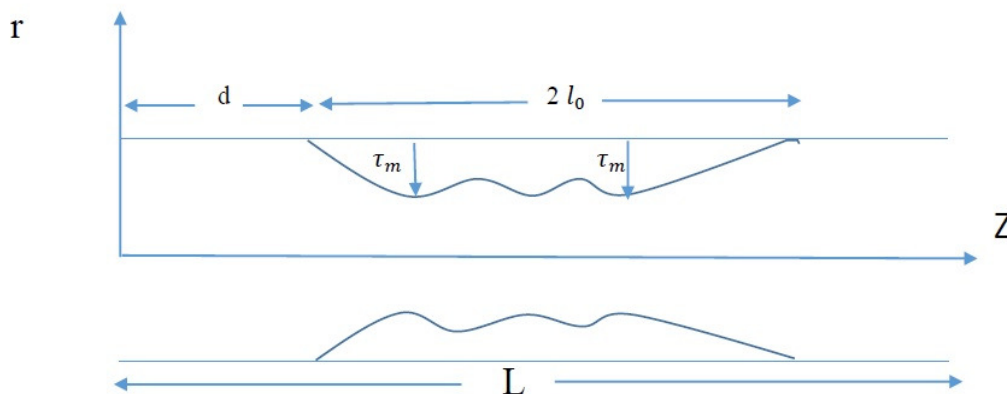We solve equation (12) for value of w by using boundary conditions (9) – (11).

Figure-1: The stenosis geometry in an artery.

Flow rate is computed using

$$Q_i^k = 2\pi \left(R_i^k\right)^2 \int_0^1 x_j w_{i,j}^k \, dx_j \qquad (13)$$

Resistance to flow (Resistive Impedance) is calculated using

$$X_i^k = \frac{\left| L\left(\frac{\partial p}{\partial z}\right)^k \right|}{Q_i^k} \qquad (14)$$

For numerical calculations following data is used.
d = 0.5 cm, $l_0 = 1 cm, L = 3 cm, a = 0.08 \, cm, \rho = 1.06 \, g/cm^3, f_p = 1.2 \, Hz,$

$\mu = 0.035 \, P, \tau_m = 0.2a, A_0 = 10 \, g \, cm^{-2} s^{-2}, A_1 = 0.2A_0, \beta = 0.1$.

Results and discussion

Using the parameters shown above, axial velocity, flow rate, resistance to flow (Impedance) is calculated numerically using MATLAB simulation.

The Figure-2 exhibits variation of axial velocity in radial direction for different times. The axial velocity increases with time. It goes on decreasing in radial direction. On the wall of artery it becomes zero due to no slip condition.
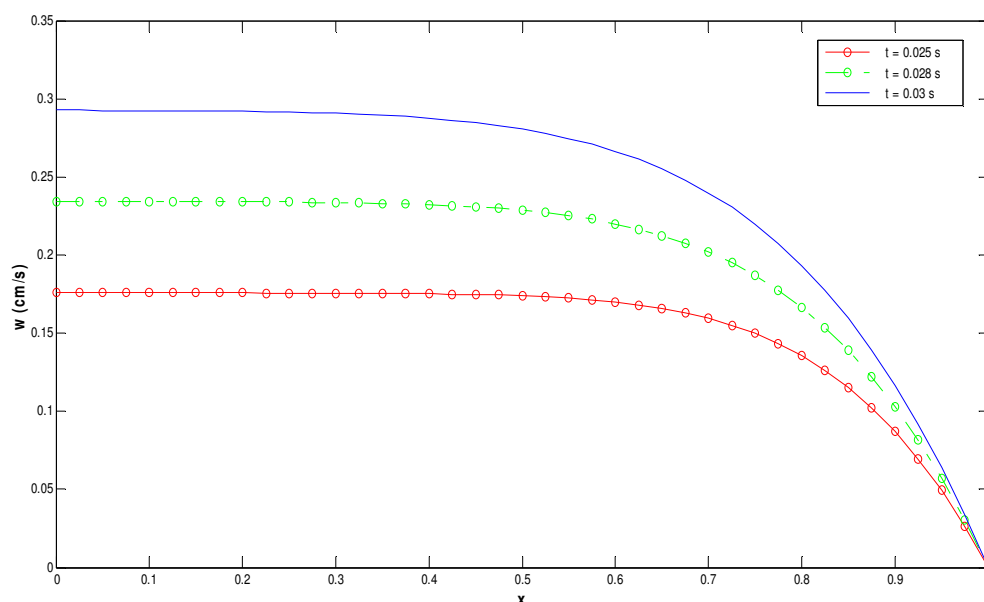

Figure-2: Variation of axial velocity in radial direction for different times.
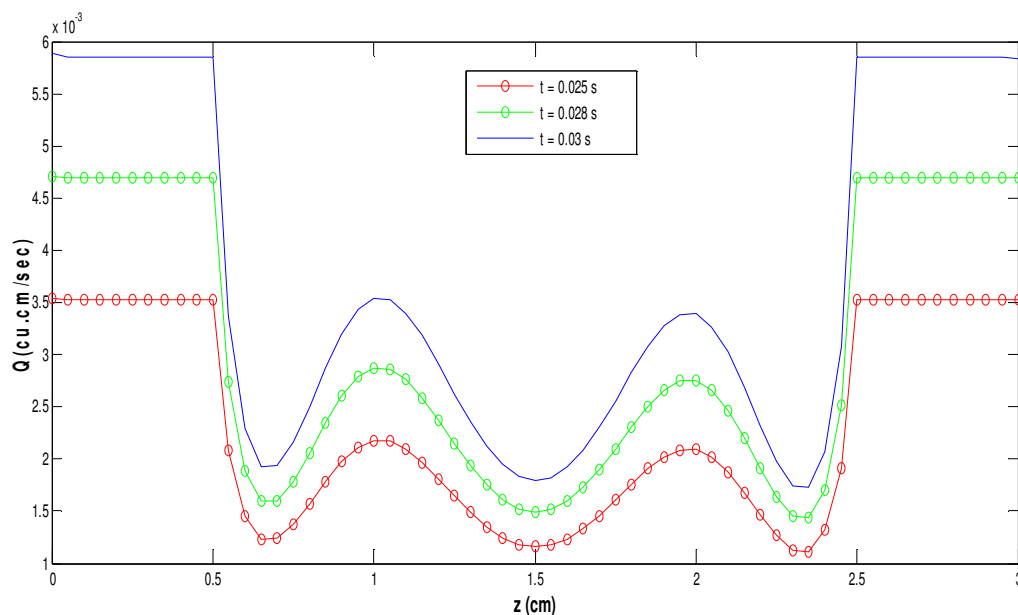

Figure-3: Flow rate distribution over length of artery for varying time.

The Figure-3 shows distribution of rate of flow along the length of the artery. It is also noted for different values of time. Flow rate remains constant before and after the stenosis region, it decreases as the height of stenosis increases and it increases as height of stenosis decreases. As time increases the flow rate also increases.

The Figure-4 shows Resistance to flow (Resistive Impedance) distribution over length of artery for varying time. As time increases the impedance decreases. Impedance follows the shape of the stenosis. It increases as height of stenosis increases and decreases with height of stenosis.

In Figure-5, rate of flow is plotted at different axial positions. Rate of flow is plotted at z = 0.25 cm, z = 1 cm, z = 2.75 cm. In the region of stenosis flow rate lower as compared to non-stenosis region. In the beginning and at the end of the artery where stenosis is not present, flow rate is same. In stenosis region it is less.
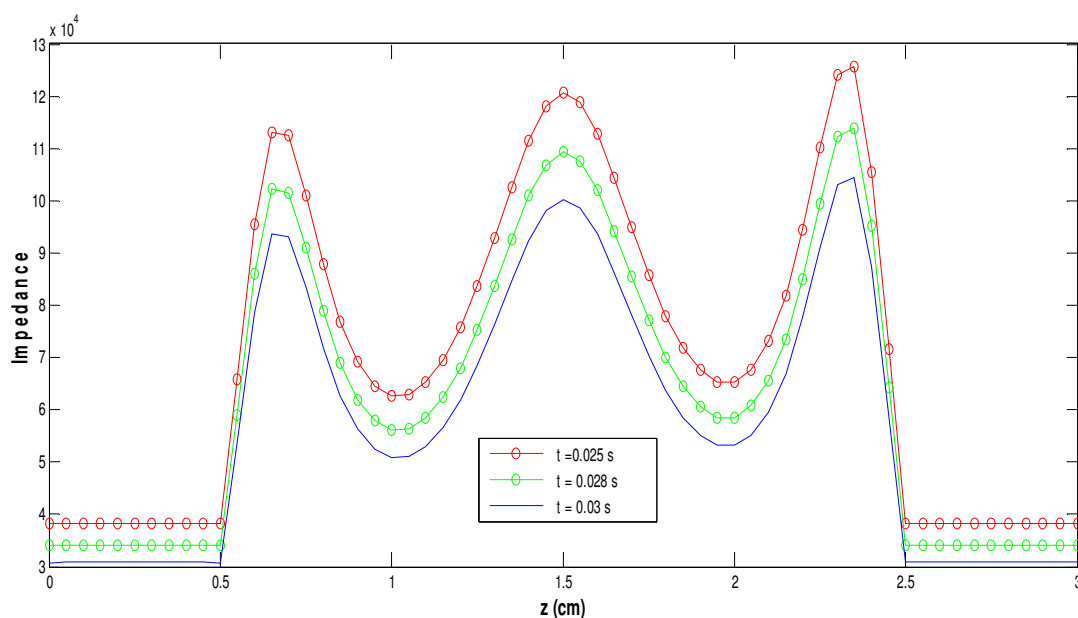

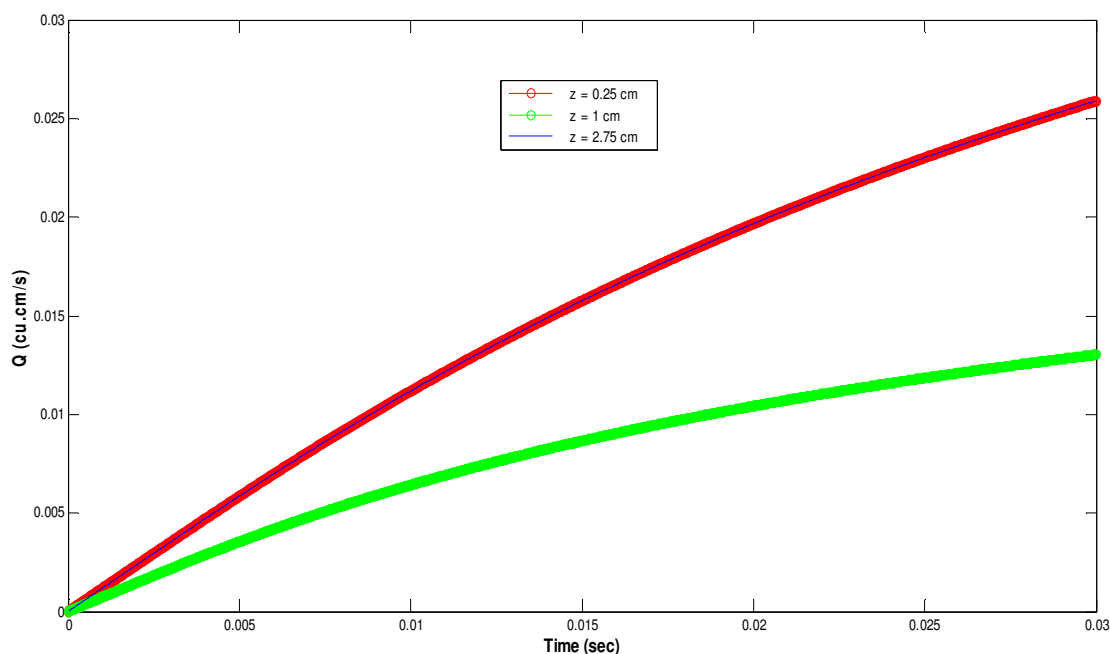**Figure-4:** Resistance to flow (Impedance) distribution over length of artery for varying time.

Figure-5: Flow rate at different axial positions.

Conclusion

Flow rate is more in normal artery than that of in stenosed region. In the region of stenosis flow rate decreases. Flow rate decreases as height of stenosis increases. There is no resistance to flow outside stenosis region but in stenosed region resistive impedance is observed. In stenosed region as height of stenosis increases, the impedance increases and it decreases as height of stenosis decreases. It follows the shape of stenosis.

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