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On Uniform Exponential Stability of Self adjoint Evolution Family: By Weak Rolewicz Approach

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Abstract

In this article we prove that if $\mathrm{P}=\{\mathrm{P}(s,t)\}_{s\geq t\geq 0}$ *is a self adjoint and strongly q-periodic continuous evolution family of bounded linear operators acting on a complex or real Hilbert space* H *then* P *is uniformly exponentially stable if for each unit vector* $x\in H$ *the integral* $\int_0^\infty \phi(\langle \mathrm{P}(s,0)x,x\rangle)\,ds$ *is bounded, where* $\phi: R_+ := [0,\infty) \to R_+$ *is a non-decreasing function such that* $\phi(0) = 0$ *and* $\phi(s) > 0$ *for all* $s\in(0\infty)$.

Keywords: EvolutionFamily, uniform exponential stability, strong rolewicz condition.

Introduction

In 2012, Constantin Buse and Gul Rahmat[1] proved a result for positive evolution family by Weak Rolewicz type approach. In fact *they proved that if* ϕ *is a non-decreasing function and* $\mathrm{P}=\{\mathrm{P}(s,t)\}_{s\geq t\geq 0}$ *is a positive and strongly q-periodic continuous evolution family of bounded linear operators acting on a complex or real Hilbert space* H *satisfying*

$$\int_0^\infty \phi(|\langle \mathrm{P}(s,0)x,x\rangle|)\,ds<\infty$$

,then the family P is uniformly exponentially stable. They asked a question whether the above result is true for self adjoint q-periodic evolution families or not. We worked on that problem and conclude that the result holds true for self adjoint q-periodic evolution families. It will be better to describe the historical background of the study before writing our result.

Datko[2] brought forth one of the important result in the stability of strongly continuous semigroup which argues that a strongly continuous semigroup $\mathrm{S}=\{\mathrm{S}(s)\}_{s\geq 0}$ of bounded linear operator sacting on complex or real Banach space is uniformly exponentially stable if and only if $\int_0^\infty \|\mathrm{S}(s)\|\,ds<\infty$.

Pazy[3] had a research on the results of Datko and further improved his attempt by stating that a strongly continuous semigroup of bounded linear operators acting on real or complex Banach space is uniformly exponentially stable if and only if $\int_0^\infty \|\mathrm{S}(s)\|^p\,ds<\infty$, for any $p\geq 1$.

Working with similar problem Rolewicz[4] generalizes the Pazy theorem a step ahead. He stated that if $\int_0^\infty \phi\|\mathrm{S}(s)\|^p\,ds<\infty$, then the semigroup S is uniformly exponentially stable. Later on special cases were proved by Zabcyzyk[5] and Przyluski[6]. Zheng[7] and Littman[8] obtained the new proofs of Rolewicz from which they discard the condition of continuity on ϕ.

Let X be a Banach space and $\overset{*}{X}$ be its dual space then $\mathrm{S}=\{\mathrm{S}(s)\}_{s\geq 0}$ is called weak L^p stable for $p\geq 1$, if

$$\int_0^\infty |\langle \mathrm{S}(s)x,x^*\rangle|^p\,ds<\infty. \tag{1}$$

It is important to note that weak L^p stability of a semigroup does not imply its uniform exponential stability, counter examples can be found in[9-11]. For further results on this topic we recommend[12-13]. Recently Constantin Buse and Gul Rahmat[1] tried to extend the result of (1) to evolution family by Weak Rolewicz type approach. Our aim is to improve the result[1] for self adjoint q-periodic evolution families.

In the first section of this article we will give some preliminaries and in second section we will present our main result.

Preliminaries

We denote by $R, C\ and\ N$ the sets of real numbers, complex numbers and the non-negative integers respectively. $\sigma(A)$ denotes the spectral radius of A and $L(X)$ the space of all bounded linear operators acting on X. As usual, $\langle .,. \rangle$ denotes the scalar product on a Hilbert space H. The norms in X, H, $L(X)$, $L(H)$ will be denoted by the same symbol $\|.\|$. A family satisfying the following properties is called q-periodic(for some $q \geq 1$) strongly continuous evolution family. i. $\mathrm{P}(s,s)=1$, ii. $\mathrm{P}(s,t)P(t,r)=\mathrm{P}(s,r)$, iii $\mathrm{P}(s+q,t+q)=\mathrm{P}(s,t)$, iv The map $(s,t)\to \mathrm{P}(s,t)x : \{(s,t):\ s,t\in R_+\ where\ s\geq t\} \to H$ is continuous for all $s \geq t$.

A family P is said to be exponentially bounded if their exists $v\in R\ and\ M_v \geq 0$ such that

$$\|\mathrm{P}(s,t)\| \leq M_v\, e^{v(s-t)}\ for\ all\ s\geq t. \tag{2.1}$$

The growth bound of exponentially bounded evolution family P is defined by $w_0(\mathrm{P}) := \inf\{v\in R : there\ is\ M_v \geq 0\ such\ that\ \|\mathrm{P}(s,t)\| \leq M_v e^{v(s-t)}\}$. The family P is uniformly exponentially stable if $w_0(\mathrm{P})<0$ .An evolution family P is called selfadjoint if each operator $\mathrm{P}(s,t)$ with $s\geq t$, is self adjoint.

Here we will recall few lemmas from the paper of Constant in Buse and Gul Rahmat[1], without proofs so that the paper will be self-contained.

Lemma 2.1 : If the spectral radius of $K\in L(X)$ is greater or equal to 1 then for all $0<\varepsilon<1$ and any sequence $(b_n)\ with\ b_n \to 0\ (as\ n\to\infty)$ and $\|(b_n)\|_\infty \leq 1$ there exists aunit vector $v_0 \in X$ such that $(1-\varepsilon).|b_n| \leq \| K^n v_0 \|\ \ for\ all\ n\in N$, where X is a complex Banach space.

Throughout this article (p_n) will denote a sequence of non-negative real numbers such that $1\leq q\leq p_{n+1}-p_n \leq \alpha\ \ for\ every\ n\in N$ and some positive real number α.

Lemma 2.2 Let $\mathrm{P}=\{\mathrm{P}(s,t)\}_{s\geq t\geq 0}$ is a q-periodic ($q\geq 1$) strongly continuous evolution family of bounded linear operators acting on a Banach space X. Suppose that (p_n) is a sequence as defined before. If $\mathrm{P}=\{\mathrm{P}(s,t)\}_{s\geq t\geq 0}$ is not uniformly exponentially stable then there exists a positive constant C with the properties that for every sequence $(k_n)\ with\ k_n \to 0\left(as\ n\to\infty\right) and \|(k_n)\|_\infty \leq 1$ there exists a unit vector $v_0 \in X$ such that

$$C\,|k_{n+1}| \leq \|\mathrm{P}(p_n,0)v_0\|\ \ for\ all\ n\in N. \tag{3}$$

An evolution family $\mathrm{P}=\{\mathrm{P}(s,t)\}_{s\geq t\geq 0}$ is said to satisfy the strong discrete Rolewicz condition if

$$\sum_{n=0}^{\infty} \phi(\|\mathrm{P}(p_n,0)x\|) < \infty. \tag{4}$$

Lemma2.3 Let $\mathrm{P}=\{\mathrm{P}(s,t)\}_{s\geq t\geq 0}$ is a strongly continuous q-periodic ($q\geq 1$) evolution family acting on X. If the family P satisfies $\sum_{n=0}^{\infty}\phi(\|\mathrm{P}(p_n,0)x\|)<\infty$ then it is uniformly exponentially stable, where ϕ is an R-function.

Results and Discussion

Let $\mathrm{P}=\{\mathrm{P}(s,t)\}_{s\geq t\geq 0}$ is a strongly continuous evolution family of bounded linear operators acting on complex Hilbert space H. When P is self adjoint $(i.e\ \mathrm{P}(s,t)=\mathrm{P}^*(s,t)\ for\ every\ s\geq t)$ then $\langle \mathrm{P}(2s,2t)x,x\rangle = \langle \mathrm{P}(s,t)\mathrm{P}(s,t)x,x\rangle = \|\mathrm{P}(s,t)x\|^2$.

The following property of self adjoint operators is very important for our results. Let U and V are two self adjoint operators, then

$$|\langle \mathrm{UV}x,y\rangle|^2 \leq \langle \mathrm{U}^2 y,y\rangle\langle \mathrm{V}^2 x,x\rangle\quad for\ all\ x,y\in H. \tag{5}$$

Before proving our result we recall Theorem 3.3[1], without proof.

Theorem 3.1 *Let ϕ is an R-function and $\mathrm{P}=\{\mathrm{P}(s,t)\}_{s\geq t\geq 0}$ is an evolution family which is positive, strongly continuous and q-periodic ($q\geq 1$) acting on a complex Hilbert space H. If*

$$J(x)=\int_0^{\infty}\phi(\langle \mathrm{P}(s,o)x,x\rangle)\,ds<\infty\quad for\ all\ \ x\in H\ \ with\ \|x\|=1,$$

then P is uniformly exponentially stable.

The problem that was left open by Constantin Buse and Gul Rahmat[1] was "whether the result holds true for self adjoint q-periodic evolution families". We tried over that problem and after simple arrangement we got the result. The outcome which has proved simply can be seen in the following lines.

Theorem3.2 *Let* ϕ *is an R-function and* $\mathrm{P}=\{\mathrm{P}(t,s)\}_{t\geq s\geq 0}$ *is an evolution family which is self adjoint, strongly continuous and q-periodic (* $q\geq 1$ *) acting on complex Hilbert space* H*. If*

$$\int_0^\infty \phi(\langle \mathrm{P}(s,o)x,x\rangle)\,ds<\infty \; for\, all\;\; x\in H \;\; with\;\; \|x\|=1,$$

then P *is uniformly exponentially stable.*

Proof: As ϕ can be considered a continuous function so applying the mean value theorem to the function $s\rightarrow\phi(\langle \mathrm{P}(2s,o)x,x\rangle)$ on the interval [nq, (n+1)q], we find $p_n(x)$ in the same interval such that

$$\frac{1}{2}\int_0^\infty \phi(\langle \mathrm{P}(s,o)x,x\rangle)\,ds=\int_0^\infty \phi(\langle \mathrm{P}(2s,o)x,x\rangle)\,ds$$

$$=q\sum_{n=0}^{\infty}\phi\,(\langle \mathrm{P}(2p_n(x),o)x,x\rangle)$$

$$\geq\sum_{n=0}^{\infty}\phi(\langle \mathrm{P}(2p_{2n}(x),o)x,x\rangle),$$

hence

$$\sum_{n=0}^{\infty}\phi(\langle \mathrm{P}(2p_{2n}(x),o)x,x\rangle)\leq\frac{1}{2}\int_0^\infty \phi(\langle \mathrm{P}(s,o)x,x\rangle)\,ds. \qquad (6)$$

Set $s_n=(2n+2)q$

$$|\langle \mathrm{P}^{\frac{1}{2}}(s_n,0)x,y\rangle|^2=|\langle \mathrm{P}^{\frac{1}{2}}(s_n,2p_{2n}(x))\mathrm{P}^{\frac{1}{2}}(2p_{2n}(x),0)x,y\rangle|^2$$

$$\leq\langle \mathrm{P}(s_n,2p_{2n}(x))y,y)\mathrm{P}(2p_{2n}(x),0)x,x\rangle$$

$$\leq Me^{4qw}\mathrm{P}(2p_{2n}(x),0)x,x\rangle,$$

hence

$$|\langle \mathrm{P}^{\frac{1}{2}}(s_n,0)x,y\rangle|^2\leq Me^{4qw}\mathrm{P}(2p_{2n}(x),0)x,x\rangle.$$

Thus for any unit vector x in H, we have

$$\frac{1}{Me^{4qw}}|\langle \mathrm{P}^{\frac{1}{2}}(s_n,0)x,x\rangle|^2\leq \mathrm{P}(2p_{2n}(x),0)x,x\rangle.$$

Since ϕ is a non-decreasing function, so we get that

$$\phi(\frac{1}{Me^{4qw}}|\langle \mathrm{P}((n+1)q,0)x,x\rangle|^2)\leq\phi(\langle \mathrm{P}(2p_{2n}(x),0)x,x\rangle).$$

Taking summation on both sides

$$\sum_{n=0}^{\infty}\phi(\frac{1}{Me^{4qw}}|\langle \mathrm{P}((n+1)q,0)x,x\rangle|^2)$$

$$\leq\sum_{n=0}^{\infty}\phi(\langle \mathrm{P}(2p_{2n}(x),0)x,x\rangle). \qquad (7)$$

From inequality (3.2) we can write

$$\sum_{n=0}^{\infty}\phi(\langle \mathrm{P}(2p_{2n}(x),0)x,x\rangle)<\infty.$$

Then (3.3) implies that

$$\sum_{n=0}^{\infty}\phi(\frac{1}{Me^{4qw}}|\langle \mathrm{P}((n+1)q,0)x,x\rangle|^2)<\infty. \qquad (8)$$

Let n+1=2mthen (3.4) can be written as

$$\sum_{m=0}^{\infty}\phi(\frac{1}{Me^{4qw}}\|\mathrm{P}(mq,0)x\|^4)<\infty.$$

Hence using Lemma 2.3 we conclude that P is uniformly exponentially stable.

Conclusion

The main objectives of this article were to give a positive answer to the question putted recently by Constant in Buse and Gul Rahmat. In fact they proved a result about the exponential stability and positive evolution family by weak Rolewicz approach, then they stated that" is it possible to prove the same result for self ad joint evolution family". In this article we conclude that the same result is still true for self ad joint evolution family.

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