ABS Solution of Linear Programming Problems with Fuzzy Variables

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Abstract

ABS method has been broadly used to solve linear and nonlinear system of equations containing a good number of variables
and constraints. In this paper, ABS method is being applied to solve linear programming problems with fuzzy variables
provided that degeneracy has been treated correctly. This can be verified by graphical method and simplex method of linear
programming problems.

Keywords: ABS algorithm, Linear programming Problems and fuzzy variables.
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Introduction

Abaffy, Broyden and Spedicato introduced ABS algorithms to
find the solution of determined or undetermined linear systems.
Later on, this algorithm has been extended to solve linear and
nonlinear equations of different nature.

One of the most important techniques among all the applied
operations research techniques is Linear Programming.

The study of linear programming had been done by so many
researchers from different point of views for more than a half
century. But still it has enough potential with which one can
develop new approach in order to best fit the real life problems
within the limitations of linear programming.

Fuzzy linear programming problems play an important role in
fuzzy modeling which can formulate in actual environment.
Tanaka et al11 first proposed the fuzzy linear programming.
Zimmerman9 developed a method to solve fuzzy linear
programming problems by using multiobjective linear
programming technique.

A method has been proposed by Campos and Verdegay3 to
solve linear programming problems containing fuzzy
coefficients in requirement vector components and in matrix
both.

Feng et al9 presented how to apply the ABS algorithm to
simplex method and the dual simplex method. Feng et al9 also
developed a method to solve linear programming problems with
fuzzy coefficients in constraints.

Lin3 discussed a method for solving fuzzy linear programming
problems based on the satisfaction degree of the constraints.

Fuzzy linear programming problems containing objective
function components as fuzzy numbers were discussed by
Zhang et al7. To solve a system containing linear equations and
linear inequalities, an algorithm known as IABS-MPVT
algorithm has been developed by Emilio Spedicato et al8.

Hamid Esmaeili et al9 presented that ABS algorithms can be
used to solve full rank linear inequalities and linear
programming problems where the number of variables is either
greater than or equal to number of inequalities.

Fuzzy linear programming problems based on fuzzy relations
was proposed by Ramik10.

Later Fuzzy linear programming problems were solved by
Nasseri11 by the technique of classical linear programming.
After that, Fuzzy linear programming problems with fuzzy
parameters were solved by Ebrahimnejad and Nessari12 by using
the complementary slackness theorem.

A new primal-dual algorithm for solving linear programming
problems with fuzzy variables was proposed by Ebrahimnejad et
al13 by using duality theorems.

Preliminaries

Following concepts will be used throughout this paper which is
based on the function principle.

A fuzzy number $\tilde{a}$ is a triangular fuzzy number denoted by $(a_1,
a_2, a_3)$, where $a_1$, $a_2$ and $a_3$ are real numbers and its membership
function is given below.
ABS Algorithm

Let \( a_1, a_2, a_3 \) and \( b_1, b_2, b_3 \) be two triangular fuzzy numbers. Then

\[
\begin{align*}
\mu_0(x) &= \begin{cases} 
\frac{(x-a_1)}{(a_2-a_1)} & \text{for } a_1 \leq x \leq a_2 \\
\frac{(a_3-x)}{(a_3-a_2)} & \text{for } a_2 \leq x \leq a_3 \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]

Let \( \bar{A} = (a_1, a_2, a_3) \) be in \( \mathbb{F}(R) \) Then

\( \bar{A} \) is called positive if \( a_i \geq 0 \), for all \( i = 1 \) to \( 3 \);

\( \bar{A} \) is called integer if \( a_i \geq 0 \), for all \( i = 1 \) to \( 3 \) are integers and

\( \bar{A} \) is called symmetric if \( a_2 - a_1 = a_3 - a_2 \).

If each element of a fuzzy vector \( b = (b_i)_{max} \) is called nonnegative real fuzzy number then it is said to be nonnegative and denoted by \( b \geq 0 \) i.e : \( b_i \geq 0 \), \( i = 1, 2, \ldots, m \).

Consider the following \( m \times n \) fuzzy linear system with nonnegative real fuzzy numbers:

\( Ax \preceq b \)

Where \( A = (a_{ij}) \) is a nonnegative crisp matrix and \( x = (x_i) \), \( b = (b_i) \) nonnegative fuzzy vectors and \( x_i, b_i \in \mathbb{F}(R) \), for all \( 1 \leq j \leq n \) and \( 1 \leq i \leq m \) where \( \mathbb{F}(R) \) is the set of all real triangular fuzzy numbers.

ABS Algorithm

Let \( x_1 \in \mathbb{R}^n \) be an arbitrary vector. Let \( H_1 \in \mathbb{R}^{n,n} \).

Compute search vector \( p_i \):

\[
p_i = H_i^T z_i
\]

(4)

Compute step size \( \alpha_i \):

\[
\alpha_i = \frac{a_i^T x_i - b_i}{p_i^T a_i}
\]

(5)

Which is well defined with regard to (3) to (4)

Compute the new approximation of the solution using

\[
x_{i+1} = x_i - \alpha_i p_i
\]

(6)

If \( i = n \) stop; \( x_{n+1} \) solve the system (1).

Let \( w_i \in \mathbb{R}^m \) be a vector arbitrary save for the condition:

\[
w_i^T H_1 a_i = 1,
\]

(7)

and to update the matrix \( H_i \):

\[
H_{i+1} = H_i - H_i a_i w_i^T H_i
\]

(8)

There are three eligible parameters Matrix \( H_1 \) and two systems of vectors \( z_i \) and \( w_i \) in the general version of the ABS algorithm. By a suitable choice of these three parameters the new algorithms or a new formulation of the classical algorithms can be created.

Numerical Example

Consider the following linear programming problem with fuzzy variables.

Max \[ \tilde{z} = 5\tilde{x}_1 \oplus 6\tilde{x}_2 \]

Subject to \[ 10\tilde{x}_1 \oplus 3\tilde{x}_2 = (48,52,48) \]

\[ 2\tilde{x}_1 \oplus 3\tilde{x}_2 = (12,18,12) \]

\[ x_{1} \geq 0, x_{2} \geq 0 \]

The above problem then reduces to

Max \[ z = 5x_1 + 6x_2 \]

Subject to \[ 10x_1 + 3x_2 \leq 52 \]

\[ 2x_1 + 3x_2 \leq 18 \]

\[ x_1 \geq 0, x_2 \geq 0 \]
Solution by Graphical Method
Feasible solutions are
\[ x_1=0, \ x_2=6, \ z =36 \]
\[ x_1=5.2, \ x_2=0, \ z =26 \]
\[ x_1=4.25, \ x_2=3.166667, \ z =40.25 \]
As \( z \) is maximum at \( x_1 =4.25, \ x_2=3.166667 \), so it’s optimal solution.

**Solution by Simple Method**

Max \( z = 5x_1 + 6x_2 \)
subject to \( 10x_1 + 3x_2 \leq 52 \)
\( 2x_1 + 3x_2 \leq 18 \)
\( x_1 \geq 0 \)
\( x_2 \geq 0 \)

Now, using dual simplex method
The solution of the problem \( (P_2) \) is \( x_1 = 17/4, \ x_2 = 19/6, \)
Max \( z_2 = 161/4. \)

**Solution by ABS Method**

\[
A = \begin{bmatrix} 10 & 2 \\ 3 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 52^T \\ 18 \end{bmatrix}
\]

\[
A^\top = \begin{bmatrix} 10 & 3 \\ 2 & 3 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

\[
z^\top H a_i = \begin{bmatrix} 10 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 3 \end{bmatrix} = 100 + 9 = 109
\]

\[
P_1 = H^\top Z_i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \end{bmatrix}
\]

\[= x_2 = x_1 - \frac{a^\top - b_i}{a_i^\top} P_i
\]

\[
= \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 10 & 3 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \end{bmatrix}
\]

\[= [0] - \begin{bmatrix} 0 - 52 \\ 100 + 9 \end{bmatrix} = \begin{bmatrix} 520/109 \\ 156/109 \end{bmatrix}
\]
\[
\begin{align*}
x_3 &= x_2 - \left[ \frac{a_1^T x_2 - b_2}{a_2^T P_2} \right] P_2 \\
&= \begin{bmatrix}
520/109 \\
156/109
\end{bmatrix} - \begin{bmatrix}
2 & 3 \\
2 & 3
\end{bmatrix} \begin{bmatrix}
520/109 \\
156/109
\end{bmatrix} \begin{bmatrix}
-18 \\
240/109
\end{bmatrix} \\
&= \begin{bmatrix}
520/109 \\
156/109
\end{bmatrix} - \begin{bmatrix}
1040 + 468 \\
-144 + 720
\end{bmatrix} \begin{bmatrix}
-72/109 \\
240/109
\end{bmatrix} \\
&= \begin{bmatrix}
520/109 \\
156/109
\end{bmatrix} + \begin{bmatrix}
1508 - 1961 \\
576
\end{bmatrix} \begin{bmatrix}
-72/109 \\
240/109
\end{bmatrix} \\
&= \begin{bmatrix}
520/109 \\
156/109
\end{bmatrix} + \begin{bmatrix}
454 \\
576
\end{bmatrix} \begin{bmatrix}
-72/109 \\
240/109
\end{bmatrix} \\
&= \begin{bmatrix}
520/109 \\
156/109
\end{bmatrix} + 0 \cdot 788194 \begin{bmatrix}
-72/109 \\
240/109
\end{bmatrix} \\
&= \begin{bmatrix}
520/109 \\
156/109
\end{bmatrix} + \begin{bmatrix}
-56 \cdot 749968/109 \\
189 \cdot 16656/109
\end{bmatrix} \\
&= \begin{bmatrix}
463.250032/109 \\
345.16656/109
\end{bmatrix} = \begin{bmatrix}
4.250000 \\
3.166667
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
x_1 &= 4 \cdot 250000 \\
x_2 &= 3 \cdot 166667
\end{align*}
\]

**Conclusion**

We have experimented the ABS method for Fuzzy Linear Programming using an extensive numerical example and the result is verified using traditional graphical and simplex methods.

If the degeneracy has been treated properly, then the feasible solution becomes optimal after \( n \) iteration given by the ABS method where \( n \) is the rank of the matrix.

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**References**

